Math 255A' Lecture 15 Notes

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1 The Banach-Stone Theorem and Compact Operators

1.1 The Banach-Stone theorem

Lemma 1.1. *Let X be a compact, Hausdorff space. Let $X \times S^1 \to M(X)$ send $(x, \alpha) \mapsto \alpha \cdot \delta_x$. This is a homeomorphism from $X \times S^1$ to $(\operatorname{ext}(B_{M(X)}), \mathrm{wk}^*)$.*

Proof. First, we show that $\alpha\delta_x$ is an extreme point. If $\alpha\delta_x = t\mu_(1-t)\nu$, then the total variation of μ or ν must be 1. So μ and ν are supported on $\{x\}$. Since $\alpha \in S^1$, we must have $\mu = \delta_x$ or $\nu = \delta_x$.

Let $\varphi(x, \alpha) = \alpha\delta_x$. Then

$$\{(x', \alpha') : |\alpha' f(x') - \alpha f(x)| < \varepsilon\} = \varphi^{-1}\left\{\mu : |\int f \, d\mu - \alpha f(x)| < \varepsilon\right\}.$$

So this is continuous.

Injectivity: If $|\alpha\delta_x| = |\alpha'\delta_{x'}|$, then $x = x'$ and $\alpha = \alpha'$.

Finally, assume that $\mu \in \operatorname{ext}(\overline{B_{M(X)}}$. Then $|\mu|$ is a regular positive Borel measure. The **support** K of $|\mu|$ is the set

$$K = \bigcap_{\substack{C \subseteq X \text{ closed} \\ |\mu|(X \setminus C) = 0}} .$$

Then $|\mu|(X \setminus K)$ (because the measure is regular).

We need to show that K is a singleton. Suppose not. Suppose that $\overline{U} \cap \overline{V} = \varnothing$, where μ has positive measure in each. Then there is an $f : X \to [0, 1]$ such that $f|_U = 0$ and $f|_V = 1$. If μ is positive, write

$$\mu = \int f \, d\mu \cdot \frac{f\mu}{\int f \, d\mu} + \int (1 - f) \, d\mu \frac{(1 - f)\mu}{\int (1 - f) \, d\mu}.$$

These two measures are different, which contradicts the fact that μ is an extreme point. For general μ, use $\mu = \frac{d\mu}{d|\mu|}|\mu|$.

This argument shows that $K = \{x\}$. This implies that $\mu = \alpha\delta_x$ for some $\alpha \in S^1$. $\square$

Theorem 1.1 (Banach-Stone)**.** *Any isometric isomorphism $C(X) \to C(Y)$ is of the form $Tf(y) = \alpha(y)f(\tau(y))$, where $\tau : Y \to X$ is a homeomorphism and $\alpha : Y \to S^1$.*

Proof. The adjoint $T^* : M(Y) \to M(X)$ resitrcts to a continuous map $(\operatorname{ext} \overline{B}_{M(Y)}, \mathrm{wk}^*) \to (\operatorname{ext} \overline{B}_{M(X)}, \mathrm{wk}^*)$. By the lemma, we have a continuous map $Y \times S^1 \to X \times S^1$. We can view $Y = Y \times \{1\} \subseteq Y \times S^1$ and same for X. Then $T^*(\delta_y) = \alpha(y) \cdot \delta_{\tau(y)}$ for some α, τ, both continuous. Moreover, τ must be invertible. Now we have

$$Tf(y) = \langle Tf, \delta_y \rangle = \langle f, T^* \delta_y \rangle = \alpha(y)f(\tau(y)),$$

as desired. $\square$

1.2 Compact operators

Let X, Y be Banach spaces.

Definition 1.1. $A \in \mathcal{B}(X, Y)$ is **compact** if any of the following equivalent statements hold:

- $\overline{A(\overline{B}_X)}$ is norm compact.
- $\overline{A(\overline{B}_X)}$ is totally bounded.
- For any bounded sequence $(x_n)_n$ in X¡ $(Tx_n)_n$ has a norm-Cauchy subsequence.

Example 1.1. If $\dim \operatorname{ran}(A) < \infty$, then A is compact.

Proposition 1.1. *If* $\dim X = \infty$, *then* Id_X *is not compact.*

Proof. Assume (towards a contradiction that B_X is compact and hence totally bounded. So there exist$x_1, \ldots, x_n \in X$ such that $\overline{B}_X \subseteq \bigcup_{i=1}^n B(X_i, 1/2)$. Then let $y \in \overline{B}_X$ and $z \in \operatorname{span}\{x_1, \ldots, x_n\}$ be such that $\|y - z\| < (1 = \varepsilon \operatorname{dist}(y, M) > 0$. Then $\|y - z\| < (1+\varepsilon) \operatorname{dist}(y - z, M)$. So $1 < (1+\varepsilon) \operatorname{dist}(\frac{y-z}{\|y-z\|} M)$; i.e.$\operatorname{dist}(\frac{y-z}{\|y-z\|} M) > 1/(1+\varepsilon) > 1/2$. $\square$

Theorem 1.2. *Let X, Y be Banach spaces, and let $A \in \mathcal{B}(X, Y)$. Then A is compact if and only if A^* is compact.*

Proof. ($\implies$): Let $(f_n)_n \in \overline{B_{Y^*}}$. Observe that

$$\|A^* f\| = \sup\{|f(Ax) : x \in \overline{B_X}\} = \|f|_{A(\overline{B_X})}\|_\infty.$$

If $f \in \overline{B_{Y^*}}$, then f is 1-Lipschitz and bounded by 1 on the compact space $\overline{A(\overline{B_X})}$. So $\{f|_{\overline{A(\overline{B_Y})}} : f \in \overline{B_{Y^*}}\}$ is norm-compact. So there is a Cauchy subsequence in $(A^* f_n)_n$ by Arzelà-Ascoli.

($\impliedby$): $A^{**}|_X = A$. $\square$

Definition 1.2. Denote $\mathcal{B}_0(X,Y)$ as the collection of compact operators $X \to Y$ and $\mathcal{B}_{00}(X,Y)$ as the collection of finite-rank operators $X \to Y$.

Proposition 1.2. *Let X, Y be Banach spaces.*

1. *$\mathcal{B}_0$ is a closed subspaces of $\mathcal{B}(X,Y)$.*
2. *Suppose $A : X \to Y$ and $B : Y \to Z$ are bounded. If either A or B is compact, then BA is compact.*

Proof. Suppose $A \in \overline{\mathcal{B}_0(X,Y)}$; we want to show that A is compact. Consider $A(\overline{B_X}) \subseteq B_\varepsilon(C(\overline{B_X}))$. For every $\varepsilon > 0$, there is a $C \in \mathcal{B}_0(X,Y)$ such that $\|A - C\|_{\text{op}} < \varepsilon$. So we can cover $A(\overline{B_X})$ with finitely many balls of radius 2ε.

Assume A is compact, then $A(\overline{B_X})$ is totally bounded, and $B(A(\overline{B_X})) \subseteq B(\overline{A(\overline{B_X})})$. $\square$

Corollary 1.1. *$\mathcal{B}_0(X)$ is an ideal in $\mathcal{B}(X)$. So $\mathcal{B}_0(X)$ is an algebra.*

Example 1.2. Let (X, Σ, μ) be a measure space, an dlet $k \in L^2(X \times X, \mu \times \mu)$. Then define the kernel operator

$$Kf(y) = \int k(x,y) f(x)\, d\mu(x).$$

Then $K \in \mathcal{B}(L^2(\mu), L^2(\mu))$, and $\|K\|_{\text{op}} \leq \|k\|_{L^2(\mu \times \mu)}$.

K is comapct because for all $\varepsilon > 0$, there exist $\varphi_1, \dots, \varphi_n \in L^2(\mu)$ and $\psi_1, \dots, \psi_n \in L^2(\mu)$ such that

$$\left\| k(x,y) - \sum_{i=1}^n \varphi_i(x), \psi_i(y) \right\|_{L^2} < \varepsilon.$$

So a finite rank approximation gives us that K is compact.

It is not always true that we can approximate by finite rank operators, but the counterexamples tend to be complicated.

Theorem 1.3. *Let X be a compact Hausdorff space. Then the space $\mathcal{B}_{00}(C(X))$ is dense in $\mathcal{B}_0(C(X))$.*

Proof. Assume $A(\overline{B_{C(X)}})$ is totally bounded. Pick $\varepsilon > 0$, and let $U_1, \dots, U_n$ be an cover X with $x_i \in U_i$. For any $f \in \overline{B_{C(X)}}$ and $x \in U_i$, we have $|Af(x) - Af(x_i)| < \varepsilon$. There exists a partition of unity: $\varphi_1, \dots, \varphi_n$ with $0 \leq \varphi_o \leq 1$ such that $\varphi_i|_{U_i^c} = 0$ ad $\sum_{i=1}^n \varphi_i = 1$. Define

$$A_\varepsilon f(x) := \sum_{i=1}^n Af(x_i) \cdot \varphi_i(x).$$

This is finite rank because it takes values in the span of the φ_i. We then have

$$|Af(x) - A_\varepsilon f(x)| \leq \sum_{i=1}^n |Af(x) - Af(x_i)|\, \varphi_i(x) < \varepsilon. \qquad \square$$